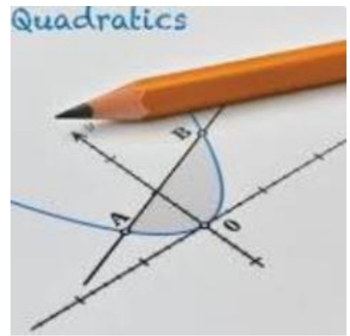
SOLVING QUADRATICS BY FACTORING

We're pretty good by now at solving equations like $2(3x - 4) + 8 = -10(x + 1)$, and you've probably had your fill of word problems which can be solved by such equations. But some of the most important applications in algebra, science, and business involve equations where the variable is *squared*. For instance, the following is called a ***quadratic equation***, due simply to the fact that the variable is being squared:



$$x^2 - 10x + 16 = 0$$

Now that's an equation!

❑ CHECKING THE SOLUTIONS OF A QUADRATIC EQUATION

First, let's verify that $\underline{x = 2}$ is a solution of the above equation (don't worry about where the 2 came from):

$x^2 - 10x + 16 = 0$ (the given equation)

$\mathbf{2}^2 - 10(\mathbf{2}) + 16 \stackrel{?}{=} 0$ (change each x into a 2)

$4 - 20 + 16 \stackrel{?}{=} 0$ (exponents and multiplication first)

$-16 + 16 \stackrel{?}{=} 0$ (add and subtract – left to right)

$0 = 0$ ✓ ($x = 2$ works!)

Fine — we have a solution: $x = 2$. Here comes the (possibly) surprising fact: This equation has another solution, namely <u>$x = 8$</u>. Watch this:

$$x^2 - 10x + 16 = 0$$

$$8^2 - 10(8) + 16 \stackrel{?}{=} 0$$

$$64 - 80 + 16 \stackrel{?}{=} 0$$

$$-16 + 16 \stackrel{?}{=} 0$$

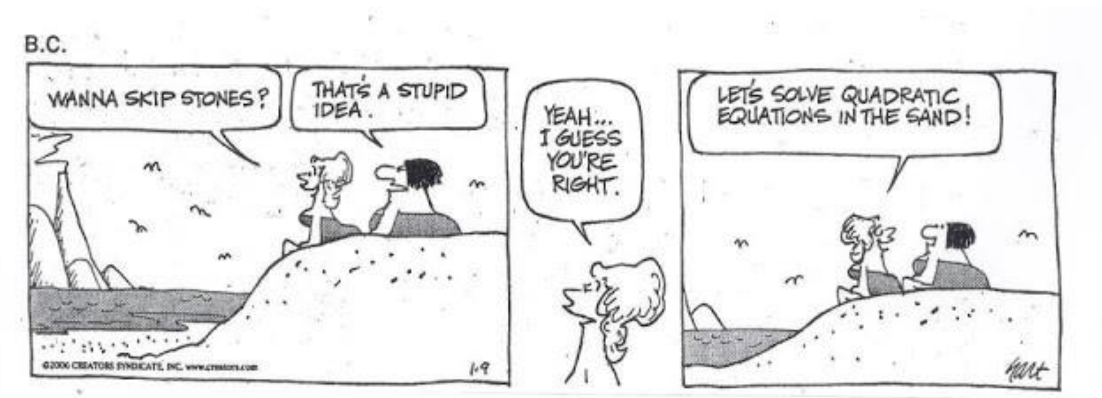
$$0 = 0 \quad \checkmark$$

One equation with <u>two</u> solutions? Yep, that's what we have. This special kind of equation, where the variable is squared (and may very well have two solutions), has a special name: We call it a ***quadratic equation*** (from "*quadrus*," Latin for a four-sided *square*).

Homework

1. For each quadratic equation, verify that the two given solutions are really solutions:

a.	$x^2 + 3x - 10 = 0$	$x = -5;\ x = 2$
b.	$n^2 - 25 = 0$	$n = 5;\ n = -5$
c.	$a^2 + 7a = -12$	$a = -3;\ a = -4$
d.	$w^2 = 7w + 18$	$w = 9;\ w = -2$
e.	$2y^2 + 8y = 0$	$y = -4;\ y = 0$



❑ *QUADRATIC EQUATIONS*

Recall the equation from part d. of the homework:

$$w^2 = 7w + 18$$

Because the variable is squared, this is an example of a ***quadratic equation***. We can write this quadratic equation in **standard form** by subtracting $7w$ from each side of the equation and then subtracting 18 from each side of the equation:

$$w^2 - 7w - 18 = 0 \quad \longleftarrow \text{ A quadratic equation in } \textbf{\textit{standard form}}$$

Notice that the quadratic (squared) term is written first, the linear term (the one with the variable to the 1st power) comes next, and the constant term comes last; the right side of the equation is 0. We also note that the coefficient of the squared term is 1, the coefficient of the linear term is –7, and the constant term is –18.

> The ***coefficient*** tells us how many of a particular thing we have. If we have $9n$, then we have 9 n's, and so the coefficient of $9n$ is 9. The coefficient of the term $-7xy$ is -7, and the coefficient of $13w^2$ is 13.
>
> Sometimes the coefficient is "implied." For example, the coefficient of u^2 is 1 (since $u^2 = 1u^2$), and the coefficient of $-m$ is -1 (since $-m = -1m$).

We can solve some quadratic equations in our head. For instance, consider the quadratic equation

$$x^2 = 25$$

What number, when squared, equals 25? Recalling the homework above, we're not surprised to learn that there are two solutions to this equation: **5** and **–5**. After all, $5^2 = 25$, and $(-5)^2 = 25$. We conclude that the solutions of the quadratic equation $x^2 = 25$ are $x = \mathbf{5}$ and $x = \mathbf{-5}$. So, as we study this and future chapters, keep in mind that a quadratic equation will likely (but not necessarily) have two different solutions.

❑ SOLVING QUADRATIC EQUATIONS BY FACTORING

Suppose we came across the statement

$$AB = 0$$

This says that the product of A and B is 0. Can we deduce anything? The only way this can occur is if either $A = 0$ **or** $B = 0$. This is the basis of the method of this chapter.

So, if we're presented with, for instance, the quadratic equation

$$2w^2 - 31w + 84 = 0$$

we use the same logic as above, but we have to first convert the left side of the equation into a multiplication problem (like AB). And to convert an expression with three terms (containing addition and subtraction) into a multiplication problem, we *FACTOR!*

> If $AB = 0$
> then $A = 0$ or $B = 0$.

Our process in solving the equation is to **factor** the quadratic expression on the left side of the equation (noting that the right side is 0), set each factor to 0, and then solve each linear equation.

<u>EXAMPLE 1:</u> **Solve the quadratic equation $\mathbf{2w^2 - 31w + 84 = 0}$.**

<u>Solution:</u>

	$2w^2 - 31w + 84 = 0$	(the original equation)
$\Rightarrow$	$(2w - 7)(w - 12) = 0$	(factor the quadratic)
$\Rightarrow$	$2w - 7 = 0$ or $w - 12 = 0$	(set each factor to 0)
$\Rightarrow$	$2w = 7$ or $w = 12$	(solve each equation)
$\Rightarrow$	$w = \frac{7}{2}$ or $w = 12$	

Thus, the solutions for w are

$$\boxed{\frac{7}{2},\ 12}$$

EXAMPLE 2: **Solve for x: $3x^2 - 7x - 40 = 0$**

Solution: This equation is in standard quadratic form, so it's all set to factor:

$$3x^2 - 7x - 40 = 0 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad (3x + 8)(x - 5) = 0 \qquad \text{(factor the left side)}$$

$$\Rightarrow \quad 3x + 8 = 0 \quad \text{or} \quad x - 5 = 0 \qquad \text{(set each factor to 0)}$$

$$\Rightarrow \quad x = \frac{-8}{3} = -\frac{8}{3} \quad \text{or} \quad x = 5 \qquad \text{(solve each equation)}$$

Thus, the final solutions to the quadratic equation are $\boxed{5, \ -\frac{8}{3}}$

It's time to practice our equation-checking. Letting $x = 5$ in the original equation $3x^2 - 7x - 40 = 0$ gives

$$3(\mathbf{5})^2 - 7(\mathbf{5}) - 40 = 3(25) - 7(5) - 40 = 75 - 35 - 40 = 0 \ \checkmark$$

Trying $x = -\frac{8}{3}$ in the original equation produces

$$3\left(-\frac{\mathbf{8}}{\mathbf{3}}\right)^2 - 7\left(-\frac{\mathbf{8}}{\mathbf{3}}\right) - 40 = 3\left(\frac{64}{9}\right) - 7\left(-\frac{8}{3}\right) - 40$$

$$= \frac{64}{3} + \frac{56}{3} - \frac{120}{3} = \frac{120}{3} - \frac{120}{3} = 0 \ \checkmark$$

EXAMPLE 3: **Solve for y: $9y^2 - 16 = 0$**

Solution: It's a good-looking quadratic (even though the middle term, the y-term, is missing), so let's factor and set the factors to zero:

$$9y^2 - 16 = 0 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad (3y + 4)(3y - 4) = 0 \qquad \text{(factor the left side)}$$

$\Rightarrow \quad 3y + 4 = 0 \quad \text{or} \quad 3y - 4 = 0$ (set each factor to 0)

$\Rightarrow \quad 3y = -4 \quad \text{or} \quad 3y = 4$ (remove the 4's)

$\Rightarrow \quad y = -\frac{4}{3} \quad \text{or} \quad y = \frac{4}{3}$ (divide both equations by 3)

Therefore, the solutions of the equation are

$\boxed{\frac{4}{3}, -\frac{4}{3}}$ which can also be written $\pm\frac{4}{3}$.

<u>EXAMPLE 4:</u> **Solve for u: $9u^2 = 42u - 49$**

<u>Solution:</u> This quadratic equation is <u>not</u> in standard form, so the first two steps will be to transform it into standard form:

$9u^2 = 42u - 49$ (the original equation)

$\Rightarrow \quad 9u^2 - 42u = -49$ (subtract $42u$)

$\Rightarrow \quad 9u^2 - 42u + 49 = 0$ (add 49 — standard form)

$\Rightarrow \quad (3u - 7)(3u - 7) = 0$ (factor)

$\Rightarrow \quad 3u - 7 = 0 \quad \text{or} \quad 3u - 7 = 0$ (set each factor to 0)

$\Rightarrow \quad u = \frac{7}{3} \quad \text{or} \quad u = \frac{7}{3}$ (solve each equation)

We obtained two solutions, but they're equal to each other, so there's just one solution (or are there two solutions which are the same?):

$\boxed{\frac{7}{3}}$

Homework

2. Solve each quadratic equation by factoring:

a. $w^2 - 12w + 35 = 0$

b. $2x^2 - 13x + 15 = 0$

c. $n^2 - 25n + 150 = 0$

d. $6a^2 - 31a + 40 = 0$

e. $x^2 + 2x - 15 = 0$

f. $y^2 - 14y + 49 = 0$

g. $z^2 - 9 = 0$

h. $3h^2 - 17h + 10 = 0$

i. $16u^2 - 8u + 1 = 0$

j. $25w^2 - 4 = 0$

3. Solve each quadratic equation by factoring:

a. $2x^2 + 10x - 28 = 0$

b. $25y^2 = 4$

c. $2z^2 + 2 = -4z$

d. $4a^2 = 3 - 4a$

e. $6u^2 = 47u + 8$

f. $0 = 4t^2 + 8t + 3$

g. $x^2 + x - 42 = 0$

h. $2n^2 + n - 3 = 0$

i. $4x^2 = 1$

j. $a^2 + 12a + 36 = 0$

k. $9k^2 = 9k - 2$

l. $9n^2 - 30n + 25 = 0$

m. $6z^2 = 10z$

n. $49t^2 = 4$

o. $25h^2 = 30h - 9$

p. $6x^2 - 7x - 10 = 0$

q. $0 = 6y^2 + y - 2$

r. $4n^2 = 25n + 21$

s. $16q^2 - 24q + 9 = 0$

t. $64t^2 - 9 = 0$

u. $-n^2 + n + 56 = 0$ [Hint: multiply each side by –1]

v. $-2x^2 + x + 3 = 0$

w. $-14a^2 = 3 - 13a$

❑ *FINDING A QUADRATIC EQUATION FROM THE SOLUTIONS*

When solving a quadratic equation by factoring, we set the quadratic to zero, factor the quadratic, set each of the two linear factors to zero, and then solve the two linear equations. However, for the next example, we have to find an equation given that we know its solutions; to do this we simply reverse the process of solving the equation.

EXAMPLE 5: **Find a quadratic equation whose solutions are 3 and –12.**

Solution:

You can use any variable you like, but I'll use x:

$x = 3$ OR $x = -12$ (the given two solutions)

$\Rightarrow\ x - 3 = 0$ OR $x + 12 = 0$ (rewrite so that each equation is 0)

$\Rightarrow\ (x - 3)(x + 12) = 0$ (the product of the factors is 0)

$\Rightarrow\ \boxed{x^2 + 9x - 36 = 0}$ (double distribute)

Note: The above example asked for "a" quadratic equation, not "the" quadratic equation. Why? Because there are an infinite number of quadratic equations which have 3 and –12 as their solutions.

For example, check out $10x^2 + 90x - 360 = 0$. If you solve it by the factoring method, you'd likely would start by factoring out the GCF of 10:

$$10x^2 + 90x - 360 = 0$$

$$\Rightarrow\ \mathbf{10}(x^2 + 9x - 36) = 0$$

$$\Rightarrow \quad 10(x-3)(x+12) = 0.$$

Dividing each side of the equation by 10 gives the equation

$$(x-3)(x+12) = 0,$$

whose solutions are exactly those required by the problem, 3 and -12. Therefore, a perfectly fine answer to the question would be the quadratic equation $10x^2 + 90x - 360 = 0$. In fact, for any number $\boldsymbol{k} \neq 0$, the quadratic equation $\boldsymbol{k}(x^2 + 9x - 36) = 0$ would properly answer the question. Nevertheless, let's agree that the best answer is the simplest one.

Homework

Find a quadratic equation (in standard form) with the given solutions:

4. 2 and -7
5. $\frac{2}{3}$ and 1
6. $-\frac{1}{2}$ and $\frac{4}{5}$
7. 0 and 9
8. The only solution is 10
9. The only solution is $-\frac{2}{3}$

Review Problems

10. Solve by factoring: $25y^2 = 1$

11. Solve by factoring: $n^2 + 16n + 64 = 0$

12. Solve by factoring: $26y^2 = 5y$

13. Solve by factoring: $25t^2 + 30t + 9 = 0$

14. Solve by factoring: $24x^2 = 14x + 3$

15. Solve by factoring: $81h^2 = 25$

16. Find a quadratic equation (in standard form) with the given solutions:

a. -3 and -5

b. $\frac{3}{2}$ and -2

c. $-\frac{3}{5}$ and $\frac{1}{2}$

d. 0 and $\frac{5}{6}$

e. The only solution is -8

f. The only solution is $\frac{7}{4}$

❑ *To ∞ and Beyond*

A. Find a quadratic equation (in standard form) with the given solutions:

1. $\pm\sqrt{7}$
2. $1+\sqrt{2}$ and $1-\sqrt{2}$
3. $-1+\sqrt{5}$ and $-1-\sqrt{5}$
4. 1 and 2 and 3

B. Solve for x: $x^3 + 3x^2 - 10x = 0$

Hint: There are three solutions.

C. Solve for n: $n^4 - 25n^2 + 144 = 0$

Hint: There are four solutions.

D. Solve for x: $\dfrac{x^2 - 5x + 6}{\sqrt[7]{\sin^2 x - \ln x + x^2 - \csc^3(e^x)}} = 0$

Solutions

1. Just plug in the proposed solutions and make sure they work.

2.

a. $5, 7$ b. $5, \frac{3}{2}$ c. $10, 15$ d. $\frac{8}{3}, \frac{5}{2}$

e. $3, -5$ f. 7 g. ± 3 h. $5, \frac{2}{3}$

i. $\frac{1}{4}$ j. $\pm\frac{2}{5}$

3.

a. $2, -7$ b. $\pm\frac{2}{5}$ c. -1 d. $\frac{1}{2}, -\frac{3}{2}$

e. $8, -\frac{1}{6}$ f. $-\frac{1}{2}, -\frac{3}{2}$ g. $6, -7$ h. $1, -\frac{3}{2}$

i. $\pm\frac{1}{2}$ j. -6 k. $\frac{1}{3}, \frac{2}{3}$ l. $\frac{5}{3}$

m. $0, \frac{5}{3}$ n. $\pm\frac{2}{7}$ o. $\frac{3}{5}$ p. $2, -\frac{5}{6}$

q. $\frac{1}{2}, -\frac{2}{3}$ r. $7, -\frac{3}{4}$ s. $\frac{3}{4}$ t. $\pm\frac{3}{8}$

u. $8, -7$ v. $-1, \frac{3}{2}$ w. $\frac{3}{7}, \frac{1}{2}$

4. $x = 2$ OR $x = -7 \Rightarrow x - 2 = 0$ OR $x + 7 = 0 \Rightarrow (x-2)(x+7) = 0$
$\Rightarrow x^2 + 5x - 14 = 0$

5. $x = \frac{2}{3}$ OR $x = -1 \Rightarrow x - \frac{2}{3} = 0$ OR $x + 1 = 0 \Rightarrow \left(x - \frac{2}{3}\right)(x+1) = 0$

$\Rightarrow x^2 + x - \frac{2}{3}x - \frac{2}{3} = 0 \Rightarrow x^2 + \frac{1}{3}x - \frac{2}{3} = 0 \Rightarrow 3x^2 + x - 2 = 0$

6. $10x^2 - 3x - 4 = 0$

7. $x^2 - 9x = 0$

8. $(x-10)(x-10) = 0 \Rightarrow x^2 - 20x + 100 = 0$

9. $9x^2 + 12x + 4 = 0$

10. $\pm\frac{1}{5}$ **11**. -8 **12**. $0, \frac{5}{26}$

13. $-\frac{3}{5}$ **14**. $\frac{3}{4}, -\frac{1}{6}$ **15**. $\pm\frac{5}{9}$

16. a. $x^2 + 8x + 15 = 0$ b. $2x^2 + x - 6 = 0$

c. $10x^2 + x - 3 = 0$ d. $6x^2 - 5x = 0$

e. $x^2 + 16x + 64 = 0$ f. $16x^2 - 56x + 49 = 0$

"Every job is a self-portrait
of the person who did it.
Autograph your work with excellence."

– Unknown